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# ON THE CONGRUENCES OF RAYS (3, 1) AND (1, 3).

By Dr. Arnold Emch, Biel, Switzerland.

1. In No. 1, Vol. 10, of the *Annals of Mathematics*, I have referred to a certain theorem concerning the projective transformations of the plane which has served Prof. Newson and myself as a means to treat the projective groups of the plane.* The theorem has been stated as follows:

*To conics $K$ and $K'$ in a plane tangent to the same straight line $l$ determine a projective transformation.*

*To find the corresponding point to a point $P$, draw from $P$ the two tangents to the conic $K$, and from the points where they intersect the line $L$ two tangents to the conic $K'$. Where these tangents intersect each other is the required point $P'$.*

In connection with the theory based upon it in the papers referred to above this theorem appears rather as an artifice and it is the purpose of this note to discuss the congruences of rays (3, 1) and (1, 3) which naturally lead to the foregoing theorem and its dualistic interpretation.

Said congruences are contained in the *Tetrahedral Complex* which, as it is well known, is formed by all rays whose four points of intersection with the four planes of a tetrahedron determine a constant anharmonic ratio. To study the relation between these congruences and the tetrahedral complex, or all the projective transformations in the plane and the complex, it seems advisable to depart from Lie's differential equation of the tetrahedral complex:

$$(b-c)\,xdydz + (c-a)\,ydzdx + (a-b)\,zdxdy = 0\,,$$

where

$$\frac{-1}{b-c} = \frac{k}{c-a} = \frac{1-k}{a-b}\,,$$

$k$ being a constant.†

In our special problem, however, we may conveniently avail ourselves of some facts about congruences of rays as they have been stated in numerous works.*

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* Arnold Emch: Projective Groups of Perspective collineations in the plane (Dissertation); Kans. Univ. Quart., Vol. IV, Nos. 2 and 4; Vol. V, No. 1. H. B. Newson: Projective Groups, Kans. Univ. Quart., Vol. IV, No. 2.

† Sophus Lie: Geometrie der Berührungs transformationen, Vol. I, page 326. Teubner, Leipzig.

2. A congruence of rays of the 3. order and 1. class, which for convenience is called a congruence (3, 1), is perfectly determined by the system of right lines connecting all the corresponding points of two collinear planes $\Sigma$ and $\Sigma'$ in space. The focal surface of this congruence is a developable surface of the third class and the fourth order and its edge of regression is a curve of the 3. order and 4. class. Designating by $A$, $B$, $C$ in $\Sigma$ and $A'$, $B'$, $C'$ in $\Sigma'$ three linear expressions in plane-coordinates which represent three corresponding points of the collineation between $\Sigma$ and $\Sigma'$, and by $\alpha$, $\beta$, $\gamma$ three arbitrary parameters, the developable surface may be considered as the envelope of the host (S-charr) of surfaces of the second class (ruled hyperboloids)

$$\alpha(BC' - B'C) + \beta(CA' - C'A) + \gamma(AB' - A'B) = 0\,, \tag{1}$$

or

$$\begin{vmatrix} A & A' & \alpha \\ B & B' & \beta \\ C & C' & \gamma \end{vmatrix} = 0\,. \tag{2}$$

Each of the equations $BC' - B'C = 0$, $CA' - C'A = 0$, $AB' - A'B = 0$, represents a surface of the second class inscribed to the developable surface of the 3. class. Taking another surface of the host (1), different from (2) when $\alpha$, $\beta$, $\gamma$ are fixed quantities, and represented by the determinant

$$\begin{vmatrix} A & A' & \alpha_1 \\ B & B' & \beta_1 \\ C & C' & \gamma_1 \end{vmatrix} = 0\,, \tag{3}$$

it can be shown that (2) and (3) have, besides being inscribed to the developable surface of the 3. class, a generatrix in common which is determined by the equations

$$\begin{vmatrix} A & \alpha & \alpha_1 \\ B & \beta & \beta_1 \\ C & \gamma & \gamma_1 \end{vmatrix} = 0\,, \quad \begin{vmatrix} A' & \alpha & \alpha_1 \\ B' & \beta & \beta_1 \\ C' & \gamma & \gamma_1 \end{vmatrix} = 0\,.$$

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* W. Fiedler: Geometrie der Lage, § 78, p. 536. Darboux: La Théorie Générale des Surfaces, Vol. II, Chapt. I.

In fact, these determinants may be written

$$A = \rho\alpha + \rho_1\alpha_1, \quad B = \rho\beta + \rho_1\beta_1, \quad C = \rho\gamma + \rho_1\gamma_1;$$

$$A' = \sigma\alpha + \sigma_1\alpha_1, \quad B' = \sigma\beta + \sigma_1\beta_1, \quad C' = \sigma\gamma + \sigma_1\gamma_1,$$

and these expressions satisfy equations (2) and (3). Conversely, it is seen that through every line of the congruence two surfaces of the second class of the host (1) may be passed. But to make the statement general, there is a single infinite number of surfaces of the host (1) passing through any line of the congruence; and they are represented by

$$\begin{vmatrix} A & A' & \alpha \\ B & B' & \beta \\ C & C' & \gamma \end{vmatrix} + \lambda \begin{vmatrix} A & A' & \alpha_1 \\ B & B' & \beta_1 \\ C & C' & \gamma_1 \end{vmatrix} = 0. \qquad (4)$$

Every line of the congruence has two focal points which are its points of tangency with the focal surface of the congruence. At these points all the surfaces of the pencil (4) are tangent to the developable surface; or the osculating planes at these points are common tangent planes to all the surfaces of the 2. class (6). From the theory of congruences it is also known that through every line of the congruence there are two developable surfaces whose generatrices belong to the congruence. In our case these surfaces become curves of the second class and they are produced by the osculating planes which pass through the line of the congruence. In fact, every osculating plane intersects the developable surface of the curve in space of the 3. order in a tangent, which is to be counted twice, and a conic. The tangents to this conic are the lines of intersection of all the osculating planes with the osculating plane containing the conic, and belong therefore to the congruence.

Through a point in space there are three osculating planes to the curve in space of the 3. order, hence also three lines of the congruence. On the other hand in each plane lies one line of the congruence. The congruence is therefore of the 3. order and 1. class. To sum up we may say:

*A congruence of rays* (3, 1) *is formed by the system of all lines of intersection between the osculating planes of a curve in space of the* 3. *order.*

*Every osculating plane intersects the developable surface in a curve of the second class, or order, and its tangents are formed by the system of intersection-lines of the given osculating plane with all the other osculating planes. Every line of the congruence is a double tangent to the developable surface and is the intersection of two osculating planes. The generatrices of the developable surface belong to the congruence, too.*

3. Assume now any two osculating planes $\Sigma$ and $\Sigma'$ of the curve of the 3. order and draw their conic-sections $K$ and $K'$ with the developable surface (Fig. 1).

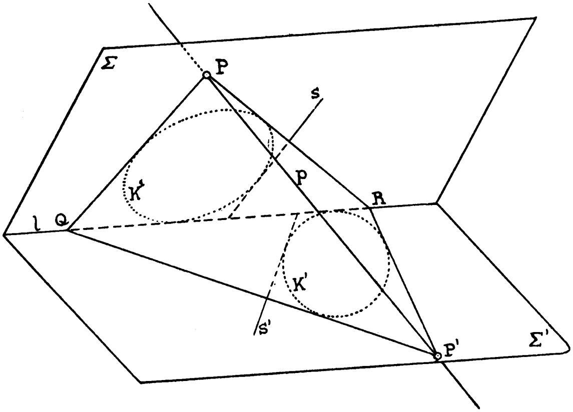

FIG. 1.

The line of intersection $l$ of $\Sigma$ and $\Sigma'$, being also a line of the congruence, is tangent to $K$ and $K'$, and so are the generatrices $s$ and $s'$ of the developable surface in $\Sigma$ and $\Sigma'$.

Taking any line $p$ of the congruence, intersecting $\Sigma$ and $\Sigma'$ in two points $P$ and $P'$, two osculating planes $PP'Q$ and $PP'R$ pass through $p$ which are also tangent to $K$ and $K'$. Through $p$ there is an infinite number of surfaces of the second class or order whose generatrices belong to the congruence and intersect $\Sigma$ and $\Sigma'$ in two collinear systems of points. Thus, $P$ and $P'$ are two corresponding points in two collinear planes $\Sigma$ and $\Sigma'$.

If we now revolve the plane $\Sigma$ into the plane $\Sigma'$ about the line $l$ as an axis we have again reached the proposition as it has been stated at the beginning.

4. For the dualistic interpretation we consider two collinear bundles (Bündel) of planes through the points $S$ and $S'$. Each two corresponding planes of these bundles intersect each other in a straight line. It is known that all these lines form a congruence of the first order and third class (1, 3), or are bi-secants of a certain curve in space of the 3. order. Choosing any two points on this curve, we can connect them successively with all lines of the congruence and obtain in this manner two collinear bundles of planes. Again, through every line of this congruence two developable surfaces may be passed whose generatrices belong to the congruence. In our case the developable surfaces are two cones of the second order which, evidently, have the ray of the congruence and the curve in space of the 3. order in common. As there are $\infty^1$ points on this curve and as the line connecting any two of these points

defines a ray of the congruence, and with this two cones of the second order which intersect each other in the curve, we may say that every curve in space of the 3. order may be considered $\infty^2$ times as the product of two intersecting cones of the second order. The rest of the intersection of these $\infty^2$ cones forms a congruence of rays (1, 3).

5. The result of the previous paragraph gives us a means to make the dualistic construction for two collinear planes when the straight line is considered as the fundamental element.

Take any ray $g$ of the congruence with the focal points $P$ and $P'$ which lie on the curve in space of the 3. order $C_3$ and construct the two cones of the second order with the vertices $P$ and $P'$ and intersecting each other in the curve $C_3$. Intersect these two cones by any plane of a general position and project the whole configuration homographically upon this plane. Without affecting the generality of the result we may assume the plane intersection of the two cones as two circles $K$ and $K'$ corresponding to the cones with the vertices $P$ and $P'$.

If we let the plane of projection coincide with the plane of the paper, Fig. 2 arises.

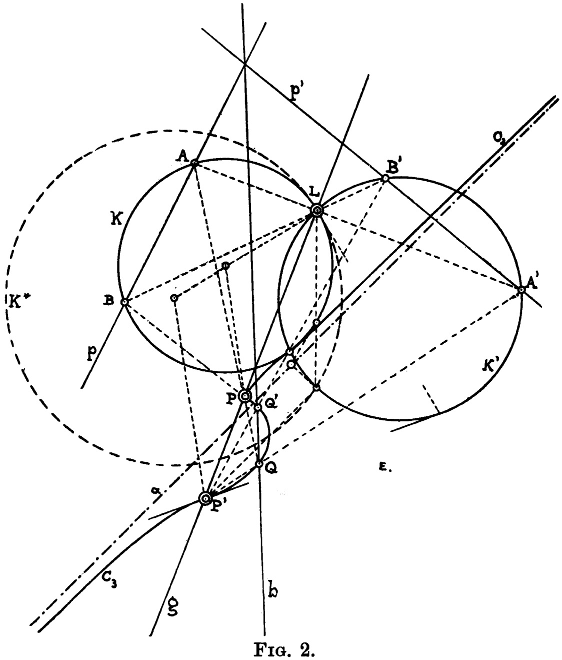

FIG. 2.

$L$ designates the point where the ray $g$ of the congruence pierces the plane of projection and is consequently one of the points of intersection of the two circles (conics) $K$ and $K'$. The construction of the curve $C_3$ is made according to well known principles in descriptive geometry and does not need to be explained here. Take any plane of the bundle through $P$ and suppose it intersects the plane of projection in the line $p$. The same plane intersects the curve $C_3$ in two points $Q$ and $Q'$ whose connection-line $h$ is a ray of the congruence. If we therefore draw a plane connecting the ray $h$ with the vertex $P'$, we shall have the corresponding plane in the collinear bundle of planes through $P'$. It intersects the plane of projection in a line $p'$, the line corresponding to the line $p$ in a collineation. The plane through $P$ and $p$ intersects the cone $(P, K)$ in two generatrices $AP$ and $BP$ on which the points $Q$ and $Q'$ are situated. To find the true location of $Q$ and $Q'$, pass two planes through the ray $g$ of the congruence and the two generatrices $AP$ and $BP$. These two planes intersect the cone $(P', K')$ in two generatrices $A'P'$ and $B'P'$. Now $AP$ and $A'P'$ lie in the same plane through $g$ and therefore intersect each other in the point $Q$ of the curve. In the same way $BP$ and $B'P'$ intersect each other in the point $Q'$ of the curve. From this we see that the construction of two points of the curve $C_3$ is identical with the construction of two collinear planes in the collinear bundles through those points. If any line $p$ in the plane of projection is given the corresponding line $p'$ in the same plane is found by connecting $A$ and $B$ (points of intersection of $p$ with $K$) with $L$ and producing these lines till they intersect the conic $K'$ in two points $A'$ and $B'$. The line connecting $A'$ with $B'$ is the corresponding line $p'$.

The dualistic theorem to the one given in § 1. is now proved :

*Two conics $K$ and $K'$ in a plane and passing through the same point $L$ determine a projective transformation in this plane.*

*To find the line $p'$ corresponding to a line $p$, connect the points of intersection $A$ and $B$ of $p$ with $K$, with $L$ and produce these lines to their points of intersection $A'$ and $B'$ with $K'$. The line connecting $A'$ with $B'$ is the required line $p'$.*[*]

6. Among the great number of special cases of collineation, perspective is of particular interest. I shall confine myself to the general case of perspective which arises when the curve $C_3$ representing the congruence (1, 3) consists of a conic $C_2$ and a straight line $g$ intersecting this conic. The congruence of bi-secants of the curve is in this case of the first order and second class (1, 2). On the ray $g$ we may again choose two points $P$ and $P'$ and pass the two cones through the conic. These cones are tangent to each other and will, therefore, intersect a plane of a general position in two tangent conics $K$ and $K'$, Fig. 3.

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[*] This theorem might be obtained immediately from the first theorem by the principle of reciprocal polars ; but it has been preferred to prove it in connection with the congruence (1, 3) as it has been pointed out in the introduction.

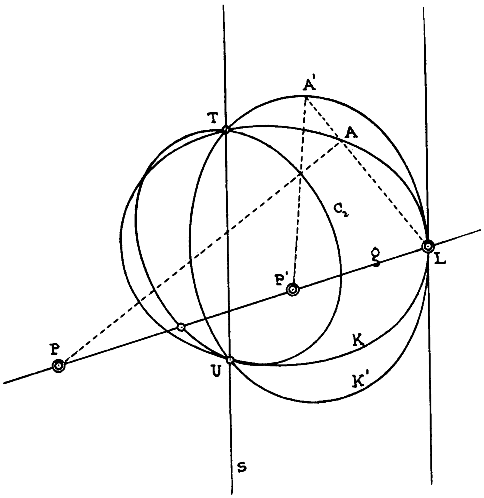

FIG. 3.

As in the general case to a line $p$ intersecting the conic $K$ to which it belongs in two points $A$ and $B$, the corresponding line $p'$ belonging to the conic $K'$ is obtained by drawing lines connecting $A$ and $B$ with $L$ which when produced intersect the conic $K'$ in two points $A'$ and $B'$. The line joining $A'$ with $B'$ is the required line $p'$, Fig. 4. From projective geometry is known *that two conics $K$ and $K'$ intersecting each other in the points $T$ and $U$ determine a perspective collineation with the line $s$ joining $T$ with $U$ as an axis and the intersection of two common tangents as a center.*

From this follows that the conics $K$ and $K'$ in Fig. 4 are in perspective with regard to the line $s$ as an axis and the point of tangency $L$ as a center.

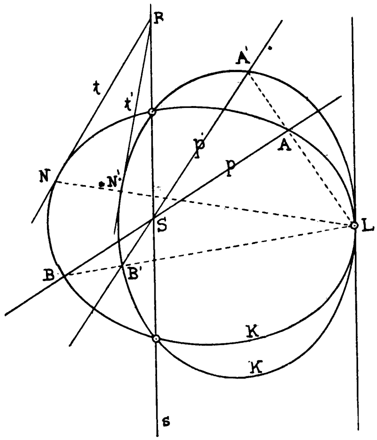

FIG. 4.

Hence, the lines $p$ and $p'$ intersect each other in a point $S$ of the axis $s$. Similarly, two corresponding tangents $t$ and $t'$ to the conics $K$ and $K'$ intersect in a point $R$ of $s$.

The dualistic interpretation of Fig. 4 leads to a theorem which has been stated in my dissertation (loc. cit.) and which reads as follows:

*Each two conics tangent to each other determine a perspective collineation with the common tangent at their point of tangency as the axis and the intersection-point of their two other common tangents as the center of collineation.* (See Fig. 5).

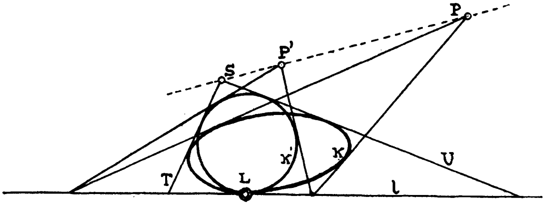

FIG. 5.

The points $P$ and $P'$ represent two collinear points, $S$ the center, and $l$ the axis of perspective collineation.

If, in Fig. 4, we assume $K$ and $K'$ as tangent circles, then the points $T$ and $U$ are the imaginary circular points, and the line $s$ is at infinity. Corresponding lines $p$ and $p'$ are all parallel. This case gives us *similarity of systems in a similar position* and is illustrated in Fig. 6.

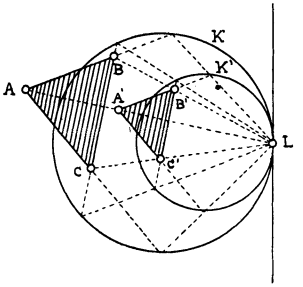

FIG. 6.

We might add a great number of such special cases and, thus, show the usefulness of studying projective transformations in the plane in connection with the congruences of rays (3, 1) and (1, 3). It is, however, easy to develop special cases from our general theorems.

UNIVERSITY OF KANSAS, *July*, 1896.